

An Upper Bound on Broadcast Subspace Codes

Yimin Pang

Department of Information Science
and Electronic Engineering
Zhejiang University
Hangzhou, China, 310027
Email: yimin.pang@zju.edu.cn

Thomas Honold

Department of Information Science
and Electronic Engineering
Zhejiang University
Hangzhou, China, 310027
Email: honold@zju.edu.cn

Abstract—Linear operator broadcast channel (LOBC) models the scenario of multi-rate packet broadcasting over a network, when random network coding is applied. This paper presents the framework of algebraic coding for LOBCs and provides a Hamming-like upper bound on (multishot) subspace codes for LOBCs.

I. INTRODUCTION

In an acyclic network where random linear network coding [1] is applied, packet transmission in a generation can be regarded as conveying the subspace spanned by the input vectors [2], and the corresponding equivalent channel model is called *linear operator channel* (LOC). We will denote the set of all i -dimensional subspaces of \mathbb{F}_q^m by $\mathcal{P}(\mathbb{F}_q^m, i)$, which is known as a Grassmannian. The overall set of subspaces of \mathbb{F}_q^m is denoted by $\mathcal{P}(\mathbb{F}_q^m)$. In general, the input and output symbols of a LOC are taken from the set of all subspaces $\mathcal{P}(\mathbb{F}_q^m)$ of \mathbb{F}_q^m (referred to as “ambient space”). If the input alphabet of a LOC happens to be $\mathcal{X} = \mathcal{P}(\mathbb{F}_q^m, l)$ (i.e. all l dimensional subspaces of \mathbb{F}_q^m) we call it a *constant dimension* LOC. If missing basis vectors of a subspace constitutes the only possible channel interference, i.e. no error vectors are inserted into the transmitted subspace, we say the LOC is *multiplicative*.

An LOC could only be viewed as either unicast channel or constant rate broadcast channel, however the benefits of network coding are mainly in packet multicast scenarios. In [3], a modified model called *linear operator broadcast channel* (LOBC) is proposed to formulate the problem of packet broadcasting over LOCs. An LOBC is a broadcast channel with subspaces as input/outputs. All receivers have their possibly different subchannel capacities and they are able to collect message of their own interests at variable rates. In [3] the capacity region of a special type of degraded *constant dimension multiplicative LOBCs* (CMLOBCs) is studied. CMLOBCs are a generalization of broadcast erasure channels. Although time sharing is sufficient to achieve the boundary of the capacity region of degraded broadcast erasure channels, the same conclusion does not hold for CMLOBCs. This motivates us to study algebraic coding theory for broadcasting over LOCs. In this paper we set up the framework of broadcast

subspace codes (i.e. a class of multishot subspace codes with unequal error protection) with separation vectors as performance parameter and prove an upper bound on the code construction.

Now we briefly summarize previous work on LOC and some related work about conventional unequal error protection codes. The concept of LOC was first proposed in [2] from a combinatorial viewpoint, and further studied in [4] and many others. Starting with [5], research on linear operator channels went into the realm of information theory. In [5] Silva et al. investigated the capacity of a random linear network coding channel with matrices as input/output symbols. Later, by regarding a LOC as a particular DMC, Uchôa-Filho and Nóbrega [6] studied the capacity of CMLOCs. Yang et al. [7], [8] computed general non-constant multiplicative LOC capacities. In [9] the rate region of multiple source access LOCs was investigated. In [3] packet broadcasting over LOC is presented, and some initial results on the capacity region of CMLOBCs are proved, while the capacity region of degraded CMLOBCs (and general LOBCs) remains unknown.

Rather than using one-shot subspace codes as investigated in [2], [4], [5], for achieving the rate region developed in [6]–[8] we need multiple uses of the same LOC, which makes it necessary to construct multishot subspace codes. Some coding strategies can be found in [10]–[12]. However, constructing good multishot subspace codes still remains a challenging problem—mainly due to the apparent lack of a nice group structure (“linearity”) on projective geometries over finite fields, see [13].

Unequal error protection (UEP) coding goes back to [14], where Masnick and Wolf suggested techniques to protect code bits in different levels. Construction and bounds on linear unequal protection codes can be found in [15] and many others. Coding and modulation issues on UEP are addressed in [16] and [17] at almost the same time.

II. CODING FOR LINEAR OPERATOR BROADCAST CHANNELS (LOBCs)

A. Broadcast Subspace Codes

Reference [3] considers the case of a multiple user LOC where a sender communicates with K receivers u_1, u_2, \dots, u_K simultaneously. The subchannels from the sender to u_k , $k = 1, 2, \dots, K$, are linear operator channels with input

and output alphabets $\mathfrak{X}, \mathfrak{Y} \subseteq \mathcal{P}(\mathbb{F}_q^m)$, where m and q are fixed. Let X, Y_1, \dots, Y_k be the corresponding random variables. The output at every receiver is taken subject to some joint transfer probability distribution $p(Y_1, Y_2, \dots, Y_k | X) = p_{Y_1, Y_2, \dots, Y_k | X}(Y_1, Y_2, \dots, Y_k | X) = p(Y_1 = Y_1, Y_2 = Y_2, \dots, Y_k = Y_k | X = X)$. Such a channel is called *Linear Operator Broadcast Channel (LOBC)*. For simplicity we restrict ourselves to a LOBC with two receivers and let $\mathfrak{M}_1, \mathfrak{M}_2$ be the alphabets of private messages for user u_1 and u_2 , respectively.

Definition 1. A broadcast (multishot) subspace code of length n for the LOBC consists of a set $\mathfrak{C} \subseteq \mathfrak{X}^n$ of codewords and a corresponding encoder/decoder pair. The LOBC encoder $\gamma: \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ maps a message pair (M_1, M_2) to a codeword $\mathbf{X} = (X_0, \dots, X_{n-1}) \in \mathfrak{C}$ (for every transmission generation). The LOBC decoder $\delta = (\delta_1, \delta_2)$ consists of two decoding functions $\delta_i: \mathfrak{Y}^n \rightarrow \mathfrak{M}_i$ ($i = 1, 2$) and maps the corresponding pair $(\mathbf{Y}_1, \mathbf{Y}_2) \in \mathfrak{Y}^n \times \mathfrak{Y}^n$ of received words to the message pair $(\hat{M}_1, \hat{M}_2) = (\delta_1(\mathbf{Y}_1), \delta_2(\mathbf{Y}_2))$.

The rate pair (R_1, R_2) , in units of q -ary symbols per subspace transmission, of the broadcast subspace code is defined as

$$R_1 = \frac{\log_q |\mathfrak{M}_1|}{n}, \quad R_2 = \frac{\log_q |\mathfrak{M}_2|}{n}. \quad (1)$$

As in [18, Ch. 14.6] we can rewrite the encoding map as

$$\gamma: (1, 2, \dots, q^{nR_1}) \times (1, 2, \dots, q^{nR_2}) \rightarrow \mathfrak{C}$$

and associate with the broadcast subspace code the parameters $((q^{nR_1}, q^{nR_2}), n)$.

B. Separation Vector for Broadcast Subspace Codes

In what follows we view $\mathcal{P}(\mathbb{F}_q^m)^n$ as a metric space relative to the *subspace distance* $d_S(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^n [\dim(X_i + Y_i) - \dim(X_i \cap Y_i)]$, where $\mathbf{X} = (X_1, X_2, \dots, X_n) \in \mathcal{P}(\mathbb{F}_q^m)^n$, $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n) \in \mathcal{P}(\mathbb{F}_q^m)^n$.

Definition 2. Let $\mathfrak{C} \subseteq \mathcal{P}(\mathbb{F}_q^m)^n$ be a multishot subspace code with (bijective) encoding map $\gamma: \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$. The *separation vector* $\mathbf{s} = (s_1, s_2) \in \mathbb{N}^2$ (\mathbb{N} is the set of positive integers) of \mathfrak{C} with respect to γ is defined as

$$\begin{aligned} s_1 &= \min_{\substack{M_1 \neq M'_1 \\ M_1, M'_1 \in \mathfrak{M}_1 \\ M_2, M'_2 \in \mathfrak{M}_2}} d_S\{\gamma(M_1, M_2), \gamma(M'_1, M'_2)\} \\ s_2 &= \min_{\substack{M_2 \neq M'_2 \\ M_1, M'_1 \in \mathfrak{M}_1 \\ M_2, M'_2 \in \mathfrak{M}_2}} d_S\{\gamma(M_1, M_2), \gamma(M'_1, M'_2)\} \end{aligned} \quad (2)$$

The separation vector is the key character to describe the error-correcting capability of a broadcast subspace code, as indicated in the following

Lemma 1. Let an LOBC encoder $\gamma: \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ have separation vector $\mathbf{s} = (s_1, s_2)$ as defined above, $\mathbf{X} = \gamma(M_1, M_2)$

the transmitted codeword, and $\mathbf{Y} \in \mathcal{P}(\mathbb{F}_q^m)^n$ the received word. Then we have:

- 1). A minimum distance decoder at user u_1 (u_2) can recover M_1 (resp. M_2) from \mathbf{Y} if $2d_S(\mathbf{X}, \mathbf{Y}) < s_1$ (resp. $2d_S(\mathbf{X}, \mathbf{Y}) < s_2$);
- 2). The minimum distance of \mathfrak{C} is $\min\{s_1, s_2\}$.

Unlike coding for LOCs, it is clear that the performance of broadcast subspace codes depends on both code and encoder map.

III. BOUNDS ON BROADCAST SUBSPACE CODES

A. Preparations

Lemma 2. Let $\gamma: \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ be an LOBC encoder with separation vector $\mathbf{s} = (s_1, s_2)$, where $s_1 < s_2$. Then there exists an auxiliary code $\mathfrak{C}_{\text{aux}} \subseteq \mathfrak{C}$ and two encoding maps $\gamma_1: \mathfrak{M}_1 \rightarrow \mathfrak{C}_{\text{aux}}$ and $\gamma_2: \mathfrak{C}_{\text{aux}} \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ such that for any $(M_1, M_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$, $\gamma_2(\gamma_1(M_1), M_2) = \gamma(M_1, M_2)$.

Proof: Without loss of generality, we assume $\mathfrak{M}_1 = \{1, 2, \dots, |\mathfrak{M}_1|\}$, $\mathfrak{M}_2 = \{1, 2, \dots, |\mathfrak{M}_2|\}$ and that the minimum distance of \mathfrak{C} is attained between two codewords of the form $\gamma(M_1, 1)$, $\gamma(M'_1, 1)$. Let $\mathfrak{C}_{\text{aux}} = \gamma(\mathfrak{M}_1, 1)$ and $\gamma_1(M_1) = \gamma(M_1, 1)$ for $M_1 \in \mathfrak{M}_1$. Since γ_1 is bijective, the map γ_2 is then uniquely defined by the condition $\gamma_2(\gamma_1(M_1), M_2) = \gamma(M_1, M_2)$. ■

Let $\mathfrak{C}_{\text{cld}}(M_2) = \gamma(\mathfrak{M}_1, M_2) = \gamma_2(\mathfrak{C}_{\text{aux}}, M_2)$ be the codeword cluster (“cloud”) corresponding to the message M_2 , so that $\mathfrak{C} = \bigsqcup\{\mathfrak{C}_{\text{cld}}(M_2); M_2 \in \mathfrak{M}_2\}$. For further discussion, we choose a unique representative codeword $\mathbf{C}_{\text{cldcen}}(M_2) \in \mathfrak{C}_{\text{cld}}(M_2)$ to denote the center of $\mathfrak{C}_{\text{cld}}(M_2)$ and let $\mathfrak{C}_{\text{cldcen}} = \{\mathbf{C}_{\text{cldcen}}(M_2) | M_2 \in \mathfrak{M}_2\}$. Additionally we choose $\mathfrak{C}_{\text{cldcen}}$ to be of minimum distance s_2 (which can clearly be done).

Lemma 3. Let $\gamma: \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ an LOBC encoder with separation vector $\mathbf{s} = (s_1, s_2)$, $s_1 < s_2$, and $\gamma_1: \mathfrak{M}_1 \rightarrow \mathfrak{C}_{\text{aux}}$ and $\gamma_2: \mathfrak{C}_{\text{aux}} \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ be defined as above. Then

- 1) The clouds $\mathfrak{C}_{\text{cld}}(M_2)$, $M_2 \in \mathfrak{M}_2$ are subspace codes of minimum distance $\geq s_1$.
- 2) The subspace distance between codewords in different clouds $\mathfrak{C}_{\text{cld}}(M_2)$ and $\mathfrak{C}_{\text{cld}}(M'_2)$, $M_2 \neq M'_2$, is at least s_2 .

Remark 1. To encode messages for a LOBC, we could start by constructing an intermediate auxiliary code $\mathfrak{C}_{\text{aux}}$ with minimum subspace distance s_1 , and “translate” the codewords of $\mathfrak{C}_{\text{aux}}$ in some way depending on the message M_2 . For example, in the case of one-shot codes ($n = 1$) we can identify \mathbb{F}_q^m with the extension field \mathbb{F}_{q^m} and use a primitive element α of \mathbb{F}_{q^m} to translate the codewords, i.e. we set $\gamma_2(\mathbf{C}, j) = \alpha^j \mathbf{C}$, where now $\mathfrak{M}_2 = \{0, 1, \dots, q^m - 2\}$ (or a suitable subset thereof). The codeword clouds $\mathfrak{C}_{\text{cld}}(M_2)$, $M_2 \in \mathfrak{M}_2$, must then be chosen subject to $d_S(\mathfrak{C}_{\text{cld}}(M_2), \mathfrak{C}_{\text{cld}}(M'_2)) = \min\{d_S(\mathbf{C}, \mathbf{C}'); \mathbf{C} \in \mathfrak{C}_{\text{cld}}(M_2), \mathbf{C}' \in \mathfrak{C}_{\text{cld}}(M'_2)\} \geq s_2$. In the special case of the “Singer cycle construction” outlined above this reduces to a condition on the auxiliary code $\mathfrak{C}_{\text{aux}}$, and the coding procedure reflects the principle of superposition coding for broadcast channel [19].

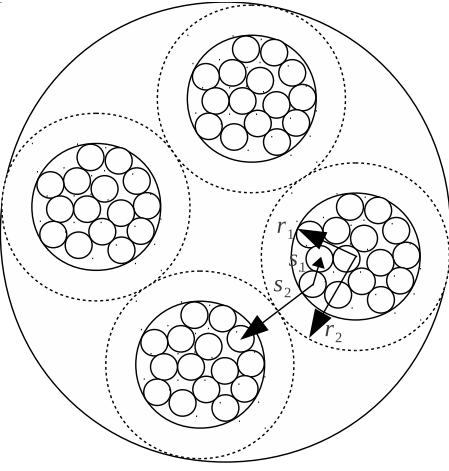


Figure 1. Sphere Packing of Broadcast Subspace Codes

The relationship between \mathfrak{C} , $\mathfrak{C}_{\text{aux}}$ and the cloud centers $\mathbf{C}_{\text{cldcen}}(M_2)$ is illustrated in Fig. 1. The overall space $\mathcal{P}(\mathbb{F}_q^m)^n$ is the ball covered by the largest sphere, the small circles denote spheres with radius $\lfloor \frac{s_1-1}{2} \rfloor$ in a fixed cluster $\mathfrak{C}_{\text{cld}}(M_2)$, and the dotted circles with radius r_2 denote larger spheres around $\mathbf{C}_{\text{cldcen}}(M_2)$ matching the error-correcting capabilities of the code $\mathfrak{C}_{\text{cldcen}}$.

B. Sphere Packing Bound for General LOBCs

The ball centered at $\mathbf{V} \in \mathcal{P}(\mathbb{F}_q^m)^n$ with radius r is defined as

$$\mathcal{B}_r(\mathbf{V}) = \{\mathbf{U} \in \mathcal{P}(\mathbb{F}_q^m)^n \mid d_S(\mathbf{U}, \mathbf{V}) \leq r\},$$

where $d_S(\mathbf{U}, \mathbf{V}) = \sum_{i=1}^n d_S(U_i, V_i)$. (We omit the symbols q and m in the expression because they are constant parameters.) The sphere centered at $\mathbf{V} \in \mathcal{P}(\mathbb{F}_q^m)^n$ with radius r is defined as

$$\mathcal{S}_r(\mathbf{V}) = \{\mathbf{U} \in \mathcal{P}(\mathbb{F}_q^m)^n \mid d_S(\mathbf{U}, \mathbf{V}) = r\}.$$

The volumes of $\mathcal{B}_r(\mathbf{V})$ and $\mathcal{S}_r(\mathbf{V})$ are defined as

$$\text{Vol}(\mathcal{B}_r(\mathbf{V})) = |\mathcal{B}_r(\mathbf{V})|$$

and

$$\text{Vol}(\mathcal{S}_r(\mathbf{V})) = |\mathcal{S}_r(\mathbf{V})|,$$

respectively. From [10] we know that the volume of $\mathcal{B}_r(\mathbf{V})$ only depends on $\mathbf{k} = (\dim V_1, \dim V_2, \dots, \dim V_n)$ and

$$\begin{aligned} \text{Vol}(\mathcal{B}_r(\mathbf{V})) &= |\mathcal{B}_r(\mathbf{k})| \\ &= \sum_{\substack{\mathbf{h} \in \{0,1,\dots,m\}^n \\ h_1+h_2+\dots+h_n \leq r}} \prod_{i=1}^n \text{Vol}(\mathcal{S}_{h_i}(k_i)), \end{aligned}$$

where

$$\text{Vol}(\mathcal{S}_{h_i}(k_i)) = \sum_{j=0}^{h_i} \binom{m-k_i}{h_i-j}_q \binom{k_i}{j}_q q^{j(h_i-j)}.$$

Since $\mathcal{B}_r(\mathbf{V})$ varies in general, we need to compute the average volume $\text{Vol}^{\text{avg}}[\mathcal{B}_r]$ of a ball with radius r in the $\mathcal{P}(\mathbb{F}_q^m)^n$ by

$$\begin{aligned} \text{Vol}^{\text{avg}}[\mathcal{B}_r] &= \frac{1}{|\mathcal{P}(\mathbb{F}_q^m)^n|} \sum_{\mathbf{V} \in \mathcal{P}(\mathbb{F}_q^m)^n} \text{Vol}(\mathcal{B}_r(\mathbf{V})) \\ &= \frac{1}{|\mathcal{P}(\mathbb{F}_q^m)^n|} \sum_{\mathbf{k} \in \{0,1,\dots,m\}^n} \binom{m}{k_1}_q \cdots \binom{m}{k_n}_q \text{Vol}(\mathcal{B}_r(\mathbf{k})) \end{aligned}$$

Definition 3. The r -neighborhood of a code $\mathfrak{S} \subseteq \mathcal{P}(\mathbb{F}_q^m)^n$ is defined as the union of all balls $\mathcal{B}_r(\mathbf{V})$ with $\mathbf{V} \in \mathfrak{S}$. The minimum volume of an r -neighborhood of $\mathfrak{S} \subseteq \mathcal{P}(\mathbb{F}_q^m)^n$ with $|\mathfrak{S}| = N$ and $d_S(\mathfrak{S}) \geq d$ is denoted by $T_n(d, N, r)$.

Theorem 1. For the parameters of an LOBC encoder $\gamma: \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ with separation vector $\mathbf{s} = (s_1, s_2)$, $s_1 < s_2$, as defined above we have the bound

$$T_n(s_1, |\mathfrak{M}_1|, \lfloor \frac{s_2-1}{2} \rfloor) \cdot |\mathfrak{M}_2| \leq |\mathcal{P}(\mathbb{F}_q^m)^n|. \quad (3)$$

Proof: The clouds $\mathfrak{C}_{\text{cld}}(M_2)$ have size $|\mathfrak{M}_1|$ and minimum subspace distance at least s_1 , and hence the volumes of their r -neighborhoods are lower-bounded by $T_n(s_1, |\mathfrak{M}_1|, r)$. Further, since clouds corresponding to different messages M_2 have distance at least s_2 , their $\lfloor \frac{s_2-1}{2} \rfloor$ -neighborhoods are pairwise disjoint. This gives (3). ■

Remark 2. The bound (3) is similar to the Hamming bound for linear binary UEP codes derived in [14], [15]. The numbers $T_n(d, N, r)$ are difficult to compute, so that we don't have an explicit bound as a function of $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$. (We can use the trivial bound $T_n(d, N, r) \geq N \cdot \text{Vol}^{\min}[\mathcal{B}_{(d-1)/2}]$ to convert (3) into such an explicit bound, which is however independent of s_2 .) Using a more elaborate argument, a lower bound (Varshamov-Gilbert like bound) can also be obtained.

C. Sphere Packing Bound for Constant Dimension LOBCs

For constant-dimension linear operator broadcast channels, the code \mathfrak{C} is a subset of the n -fold cartesian product of the Grassmannian $\mathcal{P}(\mathbb{F}_q^m, l)$ consisting of all l -dimensional \mathbb{F}_q -subspaces of \mathbb{F}_q^m , where $l \in \{0, 1, \dots, m\}$ is some fixed integer; that is $\mathfrak{C} \subseteq \mathcal{P}(\mathbb{F}_q^m, l)^n$. The sphere packing bound on constant dimension LOBCs is presented in Corollary 1, which depends on the following fact.

Fact 1. [2, Lem 4] The Gaussian coefficient $\binom{n}{l}_q$ satisfies

$$q^{l(n-l)} < \binom{n}{l}_q < 4q^{l(n-l)}$$

for $0 < l < n$.

Corollary 1. Let $\gamma: \mathfrak{M}_1 \times \mathfrak{M}_2 \rightarrow \mathfrak{C}$ be a constant-dimension LOBC encoder with separation vector $\mathbf{s} = (s_1, s_2)$, where $\mathfrak{C} \subseteq \mathcal{P}(\mathbb{F}_q^m, l)^n$ and $s_1 < s_2$, as defined above. Then

$$|\mathfrak{M}_2| < \frac{4^n q^{nl(m-l)}}{T_{n,l}(s_1, |\mathfrak{M}_1|, \lfloor \frac{s_2-1}{2} \rfloor)}, \quad (4)$$

where $T_{n,l}(d, N, r)$ is the minimum volume of an r -neighborhood in $\mathcal{P}(\mathbb{F}_q^m, l)^n$ of a constant-dimension code $\mathfrak{S} \subseteq \mathcal{P}(\mathbb{F}_q^m, l)^n$ of size N and minimum subspace distance $\geq d$.

IV. CONCLUSION

In this paper, we have converted the problem of superposition coding for LOBCs into finding multishot subspace codes with required separation vectors. We have derived sphere packing bounds for general broadcast subspace codes and for those of constant dimension. The central problem left for future work will be computing or at least bounding the volume of an r -neighborhood of $\mathfrak{S} \subseteq \mathcal{P}(\mathbb{F}_q^m)^n$. Since we lack a concept of linearity on cartesian products of projective spaces, multilevel constructions (as suggested in [16]) of broadcast subspace codes may be a useful alternative to approach the bound.

REFERENCES

- [1] T. Ho, M. Medard, R. Koetter, D. Karger, M. Effros, J. Shi, and B. Leong, "A random linear network coding approach to multicast," *Information Theory, IEEE Transactions on*, vol. 52, no. 10, pp. 4413–4430, Oct. 2006.
- [2] R. Koetter and F. Kschischang, "Coding for errors and erasures in random network coding," *Information Theory, IEEE Transactions on*, vol. 54, no. 8, pp. 3579–3591, Aug. 2008.
- [3] Y. Pang and T. Honold, "Towards the capacity region of multiplicative linear operator broadcast channels," *submitted*, 2010. [Online]. Available: <http://arxiv.org/abs/1012.5774>
- [4] D. Silva, F. Kschischang, and R. Koetter, "A rank-metric approach to error control in random network coding," *Information Theory, IEEE Transactions on*, vol. 54, no. 9, pp. 3951–3967, Sept. 2008.
- [5] D. Silva, F. Kschischang, and R. Kotter, "Communication over finite-field matrix channels," *Information Theory, IEEE Transactions on*, vol. 56, no. 3, pp. 1296–1305, Mar. 2010.
- [6] B. Uchôa-Filho and R. Nóbrega, "The capacity of random linear coding networks as subspace channels," *Arxiv preprint arXiv:1001.1021*, 2010.
- [7] S. Yang, S. Ho, J. Meng, and E. hui Yang, "Optimality of subspace coding for linear operator channels over finite fields," in *Proc. IEEE Information Theory Workshop*, 2010.
- [8] S. Yang, S. Ho, J. Meng, and E. Yang, "Linear operator channels over finite fields," *Arxiv preprint arXiv:1002.2293*, 2010.
- [9] M. Jafari, S. Mohajer, C. Fragouli, and S. Diggavi, "On the capacity of non-coherent network coding," in *Proceedings of the 2009 IEEE international conference on Symposium on Information Theory-Volume 1*, 2009, pp. 273–277.
- [10] R. Nobrega and B. Uchoa-Filho, "Multishot codes for network coding: Bounds and a multilevel construction," in *Information Theory, 2009. ISIT 2009. IEEE International Symposium on*, 2009, pp. 428–432.
- [11] R. W. Nobrega and B. F. Uchoa-Filho, "Multishot codes for network coding using rank-metric codes," in *Wireless Network Coding Conference (WiNC), 2010 IEEE*, 2010, pp. 1–6.
- [12] S. Yang, J. Meng, and E. Yang, "Coding for linear operator channels over finite fields," in *Information Theory Proceedings (ISIT), 2010 IEEE International Symposium on*. IEEE, pp. 2413–2417.
- [13] T. Etzion and A. Vardy, "Error-correcting codes in projective space," in *IEEE International Symposium on Information Theory, 2008 (ISIT 2008)*, Jul. 2008, pp. 871–875.
- [14] B. Masnick and J. Wolf, "On linear unequal error protection codes," *Information Theory, IEEE Transactions on*, vol. 13, no. 4, pp. 600–607, Oct. 1967.
- [15] I. Boyarinov and G. Katsman, "Linear unequal error protection codes," *Information Theory, IEEE Transactions on*, vol. 27, no. 2, pp. 168–175, Mar. 1981.
- [16] A. Calderbank and N. Seshadri, "Multilevel codes for unequal error protection," *Information Theory, IEEE Transactions on*, vol. 39, no. 4, pp. 1234–1248, Jul. 1993.
- [17] S. Diggavi and D. Tse, "On successive refinement of diversity," in *Proc. Allerton Conf. Communication, Control, and Computing*, p. 2004.
- [18] T. Cover and J. Thomas, "Elements of information theory," *Wiley Series In Telecommunications*, p. 542, 1991.
- [19] T. Cover, "Comments on broadcast channels," *Information Theory, IEEE Transactions on*, vol. 44, no. 6, pp. 2524–2530, Oct 1998.